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by B. Bellazzini

Let's study now an application of the wavefunction polarizations we have determined in L2. Since they are the most general overlaps, $\langle 0|\mathcal{O}_A|\kappa\,\lambda\rangle_{jm}$, and $\{|\kappa\,\lambda\rangle_{jm}\}$ form a complete set — providing we sum over all possible quantum numbers, including the mass —

$$(1)\qquad \boxed{\mathbb{1} = \sum_{\underset{\uparrow}{q}} \int_0^\infty dm^2 \sum_{j=0}^{\infty} \sum_{\sigma=-j}^{j} \int \frac{d^3\kappa}{(2\pi)^3}\frac{1}{2\kappa^0_m}\, |\kappa\,\sigma, q\rangle_{jm}\;{}_{jm}\langle \kappa\,\sigma, q|}$$

quant. numbers other than $m^2, j, \kappa^\mu, \sigma$ to uniquely specify the state

$\kappa^0_m \equiv \sqrt{\vec\kappa^2+m^2}$ $\left(d^3\Omega_\kappa \equiv \frac{d^3\kappa}{(2\pi)^3}\frac{1}{2\kappa^0_m}\right)$

Poincaré irreps (not only 1-particle states)

we are able to obtain the general structure of 2-pt functions in terms of the particles that are created/destroyed by local operators, schematically

$$\langle 0|\mathcal{O}_A^{(j_-j_+)}(x)\,\mathbb{1}\,\mathcal{O}_B^{(j'_-,j'_+)}(y)|0\rangle = \int \sum \underbrace{\langle 0|\mathcal{O}_A^{(j_-,j_+)}(x)|\kappa\,\sigma, q\rangle_{j,m}}_{\text{known up to } z\text{'s}}\;\underbrace{{}_{jm}\langle\kappa\,\sigma, q|\mathcal{O}_B^{(j'_-,j'_+)}(y)|0\rangle}_{\text{known up to } z\text{'s}}\,.$$

This way of expressing 2-pt functions in terms of the spectrum of the theory goes under the name of Källen-Lehmann spectral decomposition.

Comment on normalization:

We are working with the relativistic-invariant normalization

$$(2)\qquad {}_{jm}\langle \kappa\,\sigma, q|p\,\sigma', q'\rangle_{j'm'} = (2\pi)^4\delta^3(\vec\kappa-\vec p)\,2\kappa^0_m\,\delta_{\sigma\sigma'}\,\delta_{m^2 m'^2}\,\delta_{qq'}$$

Moreover, the resolution of the identity in (1) can be recast as

$$(3)\qquad \mathbb{1} = \sum_{q,j,\sigma}\int_0^\infty dm^2 \underbrace{\int \frac{d^4\kappa}{(2\pi)^4}\, 2\pi\cdot\delta(\kappa^2-m^2)\,\theta(\kappa^0)}_{\text{``}d\Pi^{(4)}_m(\kappa)\text{''}} \quad\longleftrightarrow\quad \mathbb{1} = \sum_{q,j,\sigma}\int \frac{d^4\kappa}{(2\pi)^4}\,\theta(\kappa^0)\,\theta(\kappa^2)$$

$d\Pi^{(4)}_m(\kappa)$ "1-body phase space"

Let's illustrate the spectral decomposition with a fully worked-out example, easily generalizable to the general case:

— Källen–Lehmann Spectral Decomposition: Vectors —

(1) $$\langle 0|\, J_\mu(x)\, J_\nu^\dagger(y)\,|0\rangle = \langle 0|\, J_\mu(x)\, \mathbb{1}\, J_\nu^\dagger(y)\,|0\rangle$$

$$= \sum_q \int dm^2 \sum_{j,\sigma} \int d\Pi^{(1)}_m(k)\, e^{-ik(x-y)} \langle 0|\, J_\mu(0)\,|k\sigma, q\rangle_{j,m} \; {}_{j,m}\langle k\sigma, q|\, J_\nu^\dagger(0)\,|0\rangle$$

Let' assume the spectrum is gapped, no massless state (except for $|0\rangle$) has overlap with J_μ's, so that we can use everything we have seen so far.

The sum over j is actually restricted to just two terms: $j=0$, $j=1$ since $A_\mu = 0 \oplus 1$:

(2) $$\langle 0|\, J_\mu(x)\, J_\nu^\dagger(y)\,|0\rangle = \sum_q \int_0^\infty dm^2 \int d\Pi^{(1)}_m(k)\, e^{-ik\cdot(x-y)} \sum_{j=0}^{1} \sum_{\sigma=-j}^{j} \langle 0|\, J_\mu(0)\,|k\sigma, q\rangle_{j,m} \cdot {}_{j,m}\langle k\sigma, q|\, J_\nu^\dagger(0)|0\rangle$$

$$= \sum_q \int_0^\infty dm^2 \int d\Pi^{(1)}_m(k)\, e^{-ik(x-y)} \Big\{ Z_{j=0}(m^2,q)\, \frac{k_\mu}{m} \cdot Z^*_{j=0}(m^2,q)\, \frac{k_\nu}{m} + \sum_\sigma Z_{j=1}(m^2,q)\, \varepsilon^\sigma_\mu(k)\, Z^*_{j=1}(m^2,q)\, \varepsilon^{*\sigma}_\nu(k) \Big\}$$

$$= \int_0^\infty dm^2 \int d\Pi^{(1)}_m(k)\, e^{-ik(x-y)} \Big\{ \sum_q \frac{|Z_{j=0}(m^2,q)|^2}{m^2}\, k_\mu k_\nu + \sum_q |Z_{j=1}(m^2,q)|^2 \underbrace{\sum_\sigma \varepsilon^\sigma_\mu(k)\, \varepsilon^{*\sigma}_\nu(k)}_{-\eta_{\mu\nu}(k) + \frac{k_\mu k_\nu}{m^2}} \Big\}$$

$$= \int_0^\infty dm^2 \int d\Pi^{(1)}_m(k)\, e^{-ik(x-y)} \Big\{ \rho_{j=0}(m^2)\, k_\mu k_\nu + \rho_{j=1}(m^2)\,(-m^2\eta_{\mu\nu} + k_\mu k_\nu) \Big\}$$

where we have defined the spectral densities $\rho_j(m^2)$:

(3) $$\boxed{\rho_{j=0}(m^2) \equiv \sum_q |Z_{j=0}(m^2,q)|^2/m^2 \geq 0\,, \qquad \rho_{j=1}(m^2) = \sum_q |Z_{j=1}(m^2,q)|^2/m^2 \geq 0}$$

In summary,

$$(4)\quad \boxed{\langle 0|\, J_\mu(x)\, J^\dagger_\nu(y)\,|0\rangle = \int dm^2 \int d\Pi^{(1)}_m(k)\, e^{-ik(x-y)} \left\{ \rho_{j=0}(m^2)\, k_\mu k_\nu + \rho_{j=1}(m^2)\left(-m^2 \eta_{\mu\nu} + k_\mu k_\nu\right)\right\}}$$

Källen-Lehmann I

Remarks:

- these spectral densities are positive, $\rho_{j=0,1}(m^2) \geq 0$. (For $\langle J_\mu J_\nu\rangle$ as opposed to $\langle J_\mu J^\dagger_\nu\rangle$, when $J^\dagger_\mu \neq J_\mu$, it would not need to be positive)
- The tensor structure is purely kinematical/geometrical (Boosted/rotated Clebsch-Gordan squared), in this example they form literally the spin-projectors $\Pi^{(j=0)}_{\mu\nu}$ and $-\Pi^{(j=1)}_{\mu\nu}$. The dynamical information about the spectrum and coupling strength is in the $\rho_j(m^2)$.
- The 1-body phase space $d\Pi^{(1)}_m(k) = \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2-m^2)\theta(k^0)$ puts all k^μ on-shell, $k^2=m^2$, so that we could have equally written it as

$$(5)\quad \boxed{\langle 0|\, J_\mu(x)\, J^\dagger_\nu(y)\,|0\rangle = \int dm^2 \int d\Pi^{(1)}_m(k)\, e^{-ik(x-y)} \left\{ \rho_{j=0}(m^2)\, k_\mu k_\nu + \rho_{j=1}(m^2)\left(-k^2 \eta_{\mu\nu} + k_\mu k_\nu\right)\right\}}$$

Källen-Lehmann I-bis

or even more explicitly, resolving the $\delta(k^2-m^2)\theta(k^0)$:

$$(6)\quad \langle 0|\, J_\mu(x)\, J^\dagger_\nu(y)|0\rangle = \int dm^2 \int \frac{d^3k}{(2\pi)^3}\frac{1}{2k^0_{(m)}}\, e^{-ik_{(m)}\cdot(x-y)} \left\{ \rho_{j=0}(m^2)\, k^{(m)}_\mu k^{(m)}_\nu + \rho_{j=1}(m^2)\left(-k^2_{(m)}\eta_{\mu\nu} + k^{(m)}_\mu k^{(m)}_\nu\right)\right\}$$

where the (m)-label in $k^{(m)}_\mu$ reminds that $k^{(m)}_0 = \sqrt{\vec{k}^2+m^2}$, i.e. $k^{(m)}_\mu k^\mu_{(m)} = m^2$

Q: what if J_μ is conserved, $\partial_\mu J^\mu = 0$?

$$\Rightarrow \partial_\mu \langle 0|\, J^\mu(x)\,|k\sigma\rangle_{jm} \propto k_\mu \left(z_{(j=0)}(m^2)\, \delta^j_0\, k^\mu + z_{(j=1)}(m^2)\, \delta^j_1\, \varepsilon^{\mu\sigma}(k)\right)$$

$$\underset{k_\mu\cdot\varepsilon^\mu=0}{\overset{!}{=}} z_{(j=0)}(m^2)\cdot m^2 = 0 \Rightarrow \boxed{z_{(j=0)}(m^2) = 0 \quad \text{removes spin-0 massive states in this case}}$$

$\Downarrow$

no $j=0$ massive state created by $J^\dagger_\mu|0\rangle$

Comment: same conclusion reached by working with (5).

Notable exception is when $m=0$ $j=0$ state exist: in that case it's better reabsorbe $1/m$ factors in $\rho_{j=0}(m^2)$ like done in (5)

which holds also for the massless case $\Rightarrow$ $\rho_{j=1}(m^2) m^2 = 0$ has $\rho_{j=1}(m^2) \propto \delta(m^2)$ as

solution, corresponding to Goldstone Bosons — $\langle 0 | J_\mu(0) | k \rangle_{j=0} \propto k_\mu$ — and spontaneous break.

— Time-ordered Källen-Lehmann —

(7) $$\langle 0 | T\, J_\mu(x)\, J_\nu^\dagger(y) | 0 \rangle = \theta(x^0 - y^0) \langle 0 | J_\mu(x) J_\nu^\dagger(y) | 0 \rangle + \theta(y^0 - x^0) \langle 0 | J_\nu^\dagger(y)\, J_\mu(x) | 0 \rangle$$

for which we need to repeat (5) for $\langle 0 | J_\nu^\dagger(y)\, J_\mu(x) | 0 \rangle$

(8) $$\langle 0 | J_\nu^\dagger(y)\, \mathbb{1}\, J_\mu(x) | 0 \rangle = \sum_q \int_0^\infty dm^2 \int d\pi_m^{(1)}(k)\, e^{+ik\cdot(x-y)} \sum_{j\sigma} \langle 0 | J_\nu^\dagger(0) | k\, \sigma, q \rangle_{jm} \cdot {}_{jm}\langle k\, \sigma, q | J_\mu(0) | 0 \rangle$$

complex conjugate up to $\mu \leftrightarrow \nu$

which are related to previous matrix elements by CPT-transformations

(9) $$\begin{cases} U_{CPT}\, J_\mu(0)\, U_{CPT}^{-1} = \eta_{CPT}^J\, J_\mu^\dagger(0) & U_{CPT}^{-1}\, J_\nu^\dagger(0)\, U_{CPT} = \eta_{CPT}^J\, J_\nu(0) \\ U_{CPT} | k\, \sigma, q \rangle_{jm} = \eta_{j,m}^{CPT} (-1)^{j-\sigma} | k\, -\sigma, \bar{q} \rangle_{jm} & U_{CPT}^{-1} | k\, \sigma, q \rangle_{jm} = \eta_{j,m}^{CPT} (-1)^{-j-\sigma} | k\, -\sigma, \bar{q} \rangle_{jm} \end{cases}$$

where the η's are CPT-phases, $|\eta|^2 = 1$, (and it turns out $\eta_{CPT}^J = -1$)

(10) $$\langle 0 | J_\nu^\dagger(0) | k\, \sigma, q \rangle_{jm} \cdot {}_{jm}\langle k\, \sigma, q | J_\mu(0) | 0 \rangle = \langle 0 | J_\nu(0) | k\, -\sigma, \bar{q} \rangle_{jm}^* \cdot {}_{jm}\langle k\, -\sigma, \bar{q} | J_\mu^\dagger(0) | 0 \rangle^*$$

and since q's and σ's are summed over in (8), $\sum_\sigma \leftrightarrow \sum_{-\sigma}$ $\sum_q \leftrightarrow \sum_{\bar{q}}$,

(11) $$\langle 0 | J_\nu^\dagger(y)\, J_\mu(x) | 0 \rangle = \int_0^\infty dm^2 \int d\pi_m^{(1)}(k)\, e^{+ik\cdot(x-y)} \Big[\overbrace{\sum_q \frac{|z_{j=0}(m^2, q)|^2}{m^2}}^{\rho_{j=0}} k_\mu k_\nu + \overbrace{\sum_q \frac{|z_{j=1}(m^2, q)|^2}{m^2}}^{\rho_{j=1}} (-\eta_{\mu\nu} k^2 + k_\mu k_\nu) \Big]$$

we get exactly the same spectral densities $\rho_{\substack{j=0 \\ j=1}}(m^2)$

(12) $$\boxed{\langle 0 | J_\nu^\dagger(y)\, J_\mu(x) | 0 \rangle = \int_0^\infty dm^2 \int d\pi_m^{(1)}(k)\, e^{+ik\cdot(x-y)} \Big[\rho_{j=0}(m^2)\, k_\mu k_\nu + \rho_{j=1}(m^2) (-\eta_{\mu\nu} k^2 + k_\mu k_\nu) \Big]}$$

We can thus get the following expression for the time-ordered correlator:

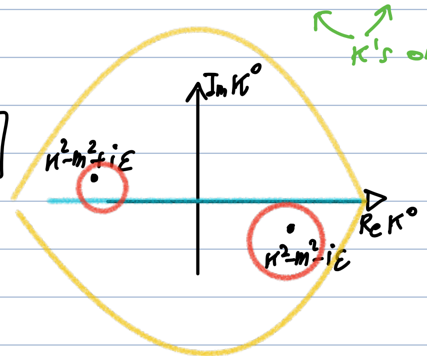
$$(13)\quad \langle 0| T J_\mu(x) J_\nu(y) |0\rangle = \int_0^\infty dm^2 \int d\Pi^{(1)}_m(K) \left(\theta(x^0-y^0)\, e^{-iK(x-y)} + \theta(y^0-x^0)\, e^{iK(x-y)}\right) \cdot$$
$$\cdot \left[\rho_{j=0}(m^2) K_\mu K_\nu + \rho_{j=1}(m^2)(-\eta_{\mu\nu}K^2 + K_\mu K_\nu)\right]$$

We can put it in a more convenient form by observing that 1st line is a sum (integral over m^2) of scalar field propagator, namely

$$(14)\quad \int d\Pi^{(1)}_m(K)\,(\quad)[\quad] = \int \frac{d^3K}{(2\pi)^3}\frac{1}{2K^0_m}\left(\theta(x^0-y^0)\, e^{-iK_{(m)}(x-y)} + \theta(y^0-x^0)\, e^{iK_{(m)}(x-y)}\right)[K \longrightarrow K_m]$$

K's on-shell

$$= \int \frac{d^4K}{(2\pi)^4}\frac{i\, e^{-iK(x-y)}}{K^2-m^2+i\varepsilon}\left[\rho_{j=0}(m^2) K_\mu K_\nu + \rho_{j=1}(m^2)(-\eta_{\mu\nu}K^2 + K_\mu K_\nu)\right]$$



K's off-shell

$\Downarrow$

Källen-Lehmann T-ordered

$$(15)\quad \boxed{\langle 0| T J_\mu(x) J_\nu(y) |0\rangle = \int_0^\infty dm^2 \int \frac{d^4K}{(2\pi)^4}\, e^{-iK(x-y)}\frac{i}{K^2-m^2+i\varepsilon}\left[\rho_{j=0}(m^2) K_\mu K_\nu + \rho_{j=1}(m^2)(-\eta_{\mu\nu}K^2 + K_\mu K_\nu)\right]}$$

Remark: it's a sum over free propagators with positive measure $\rho_j(m^2) > 0$.

Fourier transforming it

$$(16)\quad \langle 0| T J_\mu(p) J_\nu(q) |0\rangle = \int d^4x\, e^{ipx}\int d^4y\, e^{iqy}\cdot(15) = (2\pi)^4\delta^4(p+K)\,\Pi_{\mu\nu}(p)$$

$$(17)\quad \boxed{\Pi_{\mu\nu}(p) = \int dm^2 \frac{i}{p^2-m^2+i\varepsilon}\left[\rho_{j=1}(m^2)\left(-\eta_{\mu\nu}p^2 + p_\mu p_\nu\right) + \rho_{j=0}(m^2)\, p_\mu p_\nu\right]}$$

Källen-Lehmann T-ordered in momentum space

again, sum over m^2 of free-theory propagators

Now, using $\frac{1}{x \pm i\varepsilon} = P\frac{1}{x} \mp i\pi\delta(x)$ and recalling the ρ_j are real (and positive) we can extract them

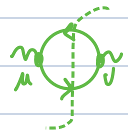
(18) $$\frac{1}{\pi}\,\mathrm{Im}\, i\Pi_{\mu\nu}(p) = \rho_{j=1}(p^2)\left(-\eta_{\mu\nu}p^2 + p_\mu p_\nu\right) + \rho_{j=0}(p^2)\, p_\mu p_\nu$$

so that ρ_j is extracted by acting with projectors

(19) $$\begin{cases} \dfrac{1}{p^4}\left(-\eta^{\mu\nu}p^2 + p^\mu p^\nu\right)\dfrac{1}{\pi}\,\mathrm{Im}\left(i\Pi_{\mu\nu}(p)\right) = \rho_{j=1}(p^2) \\[2ex] \dfrac{1}{p^4}\, p^\mu p^\nu \cdot \dfrac{1}{\pi}\,\mathrm{Im}\left(i\Pi_{\mu\nu}(p)\right) = \rho_{j=0}(p^2) \end{cases}$$

Remarks:

- The (19) give an explicit way to calculate the spectral densities, calculate the Immaginary parts of momentum space time-ordered 2-point functions, which in turn can be calculated via the optical theorem

  ⟦ In QED at one loop  $\propto \int d\Pi^{(2)}(q)\,|\mathcal{M}|^2 \propto \Gamma_{\gamma^*\to e^+e^-}$ ⟧

- If we had use (4) rather than (5), or not reabsorbed the $1/m^2$ factors in ρ_j, we would obtained an expression that differs from (15) by local terms $(a\,\eta_{\mu\nu} + b\,\partial_\mu\partial_\nu)\,\delta^4(x-y)$, since $\frac{p^2-m^2}{p^2-m^2+i\varepsilon} = 1$. This is so because T-ordering is slightly ambiguous due to product of 2-distributions, $\theta(x^0-y^0)$ and $\theta(y^0-x^0)$ with Wightman functions $\langle 0|J_\mu J_\nu|0\rangle$. Often this ambiguity is resolved by demanding extra conditions, such as Ward identities $\partial_\mu\langle T J^\mu J^\nu\rangle = 0$ in QED which is indeed satisfied in (15). However, the $\mathrm{Im}\,\Pi_{\mu\nu}$ which give rise to ρ's is free of these ambiguities. From knowing $\mathrm{Im}\,\Pi_{\mu\nu}$ one can reconstruct $\Pi_{\mu\nu}$ via dispersion relations, again leading into (17). The ambiguity in this case arise from the "number of subtractions" needed if $\rho_j(m^2) \to 0$ too slowly.

It's clear that the lessons here are general and not specific to the particular example of $\langle T J_\mu J_\nu^\dagger \rangle$. One could repeat it for $\langle 0 | T \mathcal{O} \mathcal{O}^\dagger | 0 \rangle$.

Let's move on now, and reconsider the wavefunction overlaps between fields acting on $|0\rangle$ and massless states.

— Massless particles vs Fields —

— Vectors vs Massless helicity $h = \pm 1$ —

We take $|k\ \lambda = \pm 1\rangle_{m=0}$ & A_μ and want to determine $\langle 0 | A_\mu(0) | k\ \lambda = \pm 1 \rangle_{m=0}$

We are going to see there is a clash between

(20) Lorentz-covariance vs triviality of $ISO(2)$ translations $\Rightarrow$ Gauge invariance (giving up exact covariance)

(21) $U(\Lambda) A_\mu(0) U^{-1}(\Lambda) = (\Lambda^{-1})_\mu{}^\nu A_\nu(0)$ $\qquad$ $W^{1,2} |k\ \lambda\rangle_{m=0} = 0$

(22) $\langle 0 | A_\mu(0) | \bar{k}\ \lambda = \pm 1 \rangle_{m=0} = \varepsilon_\mu^{\lambda=\pm}(\bar{k})$ $\qquad$ $\bar{k}^\mu = \begin{pmatrix} E \\ 0 \\ 0 \\ E \end{pmatrix}$

(23) $LG_{\bar{k}} = ISO(2) = $ 2D-translations $\cdot$ Rotation$_{1,2}$

2D-translations: $\left\{ \begin{array}{l} \exp(i\alpha_1 W^1 + i\alpha_2 W^2) \\ W^1 = (J^1 - K^2) E \\ W^2 = (J^2 + K^1) E \end{array} \right.$ $\qquad$ Rotation$_{1,2}$: $\exp(-i J_3 \theta)$

To show (20-21) are inconsistent with each other, let's assume them and see we run into contradiction.

Namely, let's take $\Lambda^\mu{}_\nu = W^\mu{}_\nu \in ISO(2)$ element of $LG_{\bar{k}}$

(24) $$W^\mu{}_\nu = \exp(i\alpha_1 W^1 + i\alpha_2 W^2)^\mu{}_\rho \, \exp(-iJ^3\theta)^\rho{}_\nu$$ (J^i, K^i in 4-vector irrep)

Let's start first with z-rotation only, $\Lambda^\mu{}_\nu = W^\mu{}_\nu = R_z{}^\mu{}_\nu = \exp(-iJ^3\theta)^\mu{}_\nu$

(25) $$\varepsilon^\lambda_\mu(\bar{k}) = \langle 0| A_\mu(0) |\bar{k}\,\lambda\rangle_{m=0} = \langle 0| \underbrace{U(R_z) A_\mu(0) U^{-1}(R_z)}_{(R_z^{-1})_\mu{}^\nu A_\nu(0)} \underbrace{U(R_z)|\bar{k}\,\lambda\rangle}_{|\bar{k}\,\lambda\rangle_{m=0}\exp(-i\lambda\theta)}{}_{m=0}$$

$$\Downarrow$$

(26) $$\boxed{(R_z)_\mu{}^\nu \, \varepsilon^\lambda_\nu(\bar{k}) = \varepsilon^\lambda_\mu(\bar{k})\, e^{-i\lambda\theta}}$$ eigenvalue eq. for $\varepsilon_\mu^{\lambda=\pm 1}(\bar{k})$

$$\Updownarrow$$

$$J_{3\mu}{}^\nu \, \varepsilon^\lambda_\nu(\bar{k}) = \lambda\, \varepsilon^\lambda_\mu(\bar{k})$$

$$\Downarrow$$

$$\boxed{\varepsilon^\lambda_\mu(\bar{k}) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix}}$$ $$\left[\!\left[\; J_{3\mu}{}^\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_\mu{}^\nu \;\right]\!\right]$$

Now that $\varepsilon^\lambda_\mu(\bar{k})$ are determined by z-rotations, let's see the effect of ISO(2)-translations:

(27) $$W^\mu{}_\nu = \underbrace{\exp(i\alpha_i W^i)^\mu{}_\nu}_{= T^\mu{}_\nu \text{ for ISO(2)-transl.}} = \exp \bar{E} \begin{pmatrix} 0 & \alpha_2 & -\alpha_1 & 0 \\ \alpha_2 & 0 & 0 & -\alpha_2 \\ -\alpha_1 & 0 & 0 & \alpha_1 \\ 0 & \alpha_2 & -\alpha_1 & 0 \end{pmatrix}$$

$W^1 = (J^1 - K^2)\bar{E}$
$W^2 = (J^2 + K^1)\bar{E}$

K^1-boost, K^2-boost, J^1 rot., J^2-boost

annihilate $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}^\mu$

We would like

(28) $$\varepsilon^\lambda_\mu(\bar{k}) = \langle 0| A_\mu(0) |\bar{k}\,\lambda\rangle_{m=0} = \langle 0| \underbrace{U(T) A_\mu(0) U^{-1}(T)}_{(T^{-1})_\mu{}^\nu A_\nu(0)} \underbrace{U(T)|\bar{k}\,\lambda\rangle}_{|\bar{k}\,\lambda\rangle_{m=0}}{}_{m=0}$$

$$\Downarrow$$

(28) $$\varepsilon^\lambda_\mu(k) = (T^{-1})_\mu{}^\nu \, \varepsilon^\lambda_\nu(\bar{k}) \Rightarrow \boxed{\bar{E}\begin{pmatrix} 0 & \alpha_2 & -\alpha_1 & 0 \\ \alpha_2 & 0 & 0 & -\alpha_2 \\ -\alpha_1 & 0 & 0 & \alpha_1 \\ 0 & \alpha_2 & -\alpha_1 & 0 \end{pmatrix}^\mu{}_\nu \varepsilon^{\lambda\nu}(\bar{k}) = 0}$$

from $U(W^i)|k\,\lambda\rangle = 0$ + covariance of A_μ

But this is impossible since

(29) $$\bar{E}\begin{pmatrix} 0 & \alpha_2 & -\alpha_1 & 0 \\ \alpha_2 & & & -\alpha_2 \\ -\alpha_1 & & 0 & \alpha_1 \\ 0 & \alpha_2 & -\alpha_1 & 0 \end{pmatrix}^{\mu}{}_{\nu} \begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix}^{\nu} /\sqrt{2} = \frac{\bar{E}}{\sqrt{2}}\begin{pmatrix} \alpha_2 \mp i\alpha_1 \\ 0 \\ 0 \\ \alpha_2 \mp i\alpha_1 \end{pmatrix} = \frac{\alpha_2 \mp i\alpha_1}{\sqrt{2}}\, \bar{\kappa}^{\mu} \neq 0$$

Or, going beyond linear order and exponentiating (29) (which is easy since $T\cdot\bar{\kappa}=0$)

(30) $$T^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \begin{pmatrix} \vec{\alpha}^2/2 & \alpha_2 & -\alpha_1 & -\vec{\alpha}^2/2 \\ \alpha_2 & & & -\alpha_2 \\ -\alpha_1 & & 0 & \alpha_1 \\ \vec{\alpha}^2/2 & \alpha_2 & -\alpha_1 & -\vec{\alpha}^2/2 \end{pmatrix}^{\mu}{}_{\nu} \qquad \text{with } \vec{\alpha}^2 = \alpha_1^2 + \alpha_2^2$$

(31) $$\boxed{(\text{ISO(2)-transl})^{\mu}{}_{\nu}\, \varepsilon^{\nu\,\lambda=\pm1}(\bar{\kappa}) = \varepsilon^{\mu\,\lambda=\pm1}(\bar{\kappa}) + \frac{\alpha_2 \mp i\alpha_1}{\sqrt{2}}\, \bar{\kappa}^{\mu}}$$ in contradiction with (28)

Main Lesson:

- Lorentz-covariance incompatible with A_μ creating $h=\pm1$ massless states
- However, the (31) suggests the simple fix: require Lorentz-covariance up to gauge transformations

(32) $$\boxed{U(\Lambda)\, A^{\mu}(0)\, U^{-1}(\Lambda) = (\Lambda^{-1})^{\mu}{}_{\nu}\left(A^{\nu}(0) + \partial^{\nu}\Omega\right)}$$

Indeed, repeating previous steps for $\Lambda^{\mu}{}_{\nu} = W^{\mu}{}_{\nu} = T^{\mu}{}_{\rho}\, R_z{}^{\rho}{}_{\nu} \in LG_{\bar{\kappa}} = ISO(2)$

(33) $$\varepsilon^{\lambda}_{\mu}(\bar{\kappa}) = \langle 0|A_\mu(0)|\bar{\kappa}\,\lambda\rangle_{m=0} = (W^{-1})_{\mu}{}^{\nu}\left(\langle 0|A^{\nu}(0)|\kappa\,\lambda\rangle_{m=0} + \langle 0|\partial_\mu\Omega|\bar{\kappa}\lambda\rangle_{m=0}\right) e^{-i\lambda\theta}$$

which on the $\varepsilon^{\pm}_{\mu}(\lambda)$ ±-eigenstates of J^3 gives now

(34) $$\underbrace{T_{\mu}{}^{\nu}\, \varepsilon^{\lambda}_{\nu}(\bar{\kappa})}_{=\ \varepsilon^{\lambda}_{\mu}(\bar{\kappa}) + \frac{\alpha_2\mp i\alpha_1}{\sqrt{2}}\bar{\kappa}_\mu}\, \cancel{e^{-i\lambda\theta}} = \left(\varepsilon^{\lambda}_{\mu}(\kappa) - i\bar{\kappa}_\mu \langle 0|\Omega(0)|\bar{\kappa}\,\lambda\rangle_{m=0}\right) \cancel{e^{-i\lambda\theta}}$$

which is now perfectly compatible since $\langle 0|\partial_\mu\Omega|\bar{\kappa}\,\lambda\rangle_{m=0} \propto \bar{\kappa}_\mu \langle 0|\Omega|\bar{\kappa}\,\lambda\rangle_{m=0}$

(this defines basically $\langle 0|\Omega(0)|\bar{\kappa}\,\lambda\rangle_{m=0} = \frac{i}{\sqrt{2}}(\alpha_2 \mp i\alpha_1)$)

Hence, we can now look at generic $\varepsilon_\mu^\lambda(k) = \langle 0|A_\mu(0)|k\,\lambda\rangle_{m=0}$

recalling that $|k\,\lambda\rangle_{m=0} = U(L(k,\bar{k}))\,|\bar{k}\,\lambda\rangle_{m=0}$

(35) $$\varepsilon_\mu^\lambda(k) = \langle 0|A_\mu(0)|k\,\lambda\rangle_{m=0} = L_\mu{}^\nu(\bar{k},k)\Big(\underbrace{\langle 0|A_\nu(0)|\bar{k}\,\lambda\rangle}_{\varepsilon_\nu^\lambda(\bar{k})} + \underbrace{\langle 0|\partial_\nu\Omega|\bar{k}\,\lambda\rangle}_{*\,\bar{k}_\nu}\Big)$$

$\Downarrow$ $L\cdot\bar{k} = k$

(36) $$\boxed{\varepsilon_\mu^\lambda(k) = L_\mu{}^\nu(k,\bar{k})\,\varepsilon_\nu^\lambda(\bar{k}) + *\,k_\mu}$$

Differ by covariant-ε by a gauge transformation

Likewise, acting with generic Λ

(37) $$\varepsilon_\mu^\lambda(k) = \langle 0|A_\mu(0)|k\,\lambda\rangle_{m=0} = \langle 0|U^{-1}(\Lambda)\underbrace{U(\Lambda)A_\mu(0)U^{-1}(\Lambda)}_{(\Lambda^{-1})_\mu{}^\nu(A_\nu+\partial_\nu\Omega)}\underbrace{U(\Lambda)|k\,\lambda\rangle}_{|\Lambda k\,\lambda\rangle\,e^{-i\lambda\theta_\Lambda}}{}_{m=0}$$

(38) $$\boxed{\Lambda_\mu{}^\nu\,\varepsilon_\nu^\lambda(k) = \left(\varepsilon_\mu^\lambda(\Lambda k) + *\,(\Lambda\cdot k)_\mu\right)e^{-i\lambda\theta_\Lambda}}$$

This is no surprise, after all, since $\varepsilon_\mu^\lambda(k)\,k^\mu = 0$ admits always $\varepsilon_\mu^\lambda \sim k_\mu$ since $k^2=0$

$$\Big(\varepsilon_\mu^\lambda(k)\,k^\mu = \varepsilon_\mu^\lambda(k)\,L^\mu{}_\nu(k,\bar{k})\,\bar{k}^\nu = (L^{-1})_\nu{}^\mu\,\varepsilon_\mu^\lambda(k)\,\bar{k}^\nu \underset{(36)}{=} (\varepsilon_\nu^\lambda(\bar{k}) + *\,\bar{k}_\nu)\,\bar{k}^\nu = \varepsilon_\nu^\lambda(\bar{k})\,\bar{k}^\nu \underset{k^2=0}{=} 0\Big)$$

— Maxwell equations —

What we have seen implies (free) Maxwell equation, $\langle 0|\partial_\mu F^{\mu\nu}|k\,\lambda\rangle_{m=0} = 0$

as soon as $\langle 0|F_{\mu\nu}|k\,\lambda\rangle_{m=0} \neq 0$ for (any) 2-tensor antisymmetric $F_{\mu\nu}$.

Indeed, $(*dF)^\sigma = \partial_\rho F_{\mu\nu}\,\varepsilon^{\mu\nu\rho\sigma} = (1/2, 1/2)$ exactly — not just up to gauge —

(39) $$\begin{cases} \partial_\rho \langle 0|F_{\mu\nu}|k\,\lambda\rangle_{m=0}\,\varepsilon^{\mu\nu\rho\sigma} = 0 \\ \langle 0|F_{\mu\nu}|k\,\lambda\rangle_{m=0} = \partial_{[\mu}\langle 0|A_{\nu]}|k\,\lambda\rangle_m \propto k_{[\mu}\varepsilon_{\nu]}^\lambda \end{cases} \Rightarrow \boxed{\partial_\mu\langle 0|F^\mu{}_\nu|k\,\lambda\rangle \propto k^2\varepsilon_\nu - k_\nu(k\cdot\varepsilon) = 0}$$

Maxwell eq. can't be changed

— $h_{\mu\nu} = (0,0) \oplus (1,1)$ & Gravitons —

Q: can a symmetric Lorentz 2-tensor $h_{\mu\nu}$ interpolate a massless $|\lambda|=2$ state?
A: No, only if $h_{\mu\nu}$ is 2-tensor up to gauge transformations (linear diff's)

It's enough to repeat the reasoning of A_μ, now on each Lorentz index

$$(40) \quad \varepsilon^{\lambda=\pm2}_{\mu\nu}(\bar{k}) = \langle 0| h_{\mu\nu}(0) |\bar{k}\, \lambda=\pm2\rangle_{m=0} = \langle 0| U(R_z) h_{\mu\nu}(0)\, U(R_z^{-1}) U(R_z) |\bar{k}\, \lambda\rangle_{m=0} \qquad \bar{k}^\mu = \bar{E}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$
$$= (R_z^{-1})_\mu{}^\rho (R_z^{-1})_\nu{}^\sigma \langle 0| h_{\rho\sigma}(0) |\bar{k}\,\lambda\rangle_{m=0}\, e^{-i\lambda\theta}$$

$$(41) \quad (R_z)_\mu{}^\rho (R_z)_\nu{}^\sigma \varepsilon^{\lambda=\pm2}_{\rho\sigma}(\bar{k}) = \varepsilon^{\lambda=\pm2}_{\mu\nu}\, e^{-i\lambda\theta}\Big|_{\lambda=\pm2} \Rightarrow \begin{cases} \varepsilon^{\lambda=+2}_{\mu\nu}(\bar{k}) = \varepsilon^{\lambda=+1}_\mu(\bar{k})\, \varepsilon^{\lambda=+1}_\nu(\bar{k}) \\ \varepsilon^{\lambda=-2}_{\mu\nu}(\bar{k}) = \varepsilon^{\lambda=-1}_\mu(\bar{k})\, \varepsilon^{\lambda=-1}_\nu(\bar{k}) \end{cases}$$

(notice that $\varepsilon^{\mu\,\lambda}_\mu = 0$ since $R_\mu{}^\rho R_\nu{}^\sigma \eta^{\mu\nu} = \eta^{\rho\sigma}$ i.e. $\varepsilon^\mu_\mu = \varepsilon^\mu_\mu e^{-i\lambda\theta}$)
the trace would not vanish if we were interested in $\lambda=0$ $\quad(\varepsilon^{\lambda\,\mu}_\mu = 0)$

But then, performing now 2D-translations in $LG_{\bar{k}} = ISO(2)$ like before we clash

$$(42) \quad T_\mu{}^\rho\, T_\nu{}^\sigma\, \varepsilon^{\lambda=\pm2}_{\rho\sigma}(\bar{k}) = \varepsilon^{\lambda=\pm2}_{\mu\nu}(\bar{k}) \quad \text{vs} \quad \begin{cases} \varepsilon^\lambda_{\mu\nu}(\bar{k}) = \varepsilon^\lambda_\mu(\bar{k})\, \varepsilon^\lambda_\nu(\bar{k}) \\ T_\mu{}^\nu \varepsilon^\lambda_\nu(\bar{k}) = \varepsilon^\lambda_\mu + \#\, \bar{k}_\mu \end{cases}$$

incompatible

$$\Downarrow$$

$$(43) \quad \boxed{T_\mu{}^\rho\, T_\nu{}^\sigma\, \varepsilon^\lambda_{\rho\sigma}(\bar{k}) = \varepsilon^\lambda_{\mu\nu}(\bar{k}) + \#\, \bar{k}_{(\mu}\varepsilon^\lambda_{\nu)} + \#\, \bar{k}_{(\mu}\bar{k}_{\nu)}}$$

Like before, the solution is making $h_{\mu\nu}$ 2-tensor up to gauge transformations

$$(44) \quad \boxed{U(\Lambda)\, h_{\mu\nu}(0)\, U(\Lambda^{-1}) = (\Lambda^{-1})_\mu{}^\rho (\Lambda^{-1})_\nu{}^\sigma \left(h_{\rho\sigma}(0) + \partial_{(\rho}\Omega_{\sigma)} \right)}$$

2-tensor up to gauge (linearized diffs)

since (42) gets replaced by

$$(45) \quad T_\mu{}^\rho\, T_\nu{}^\sigma\, \varepsilon^{\lambda=\pm2}_{\rho\sigma}(\bar{k}) = \varepsilon^{\lambda=\pm2}_{\mu\nu} + \langle 0| \partial_{(\mu}\Omega_{\nu)} |\bar{k}\,\lambda\rangle_{m=0}$$

↳ this can match (43) for $\Omega_\nu = A_\mu + \partial_\mu\Omega$

<u>Main lesson</u>: We see emergence of GR, from consistency of quantum theory with $m=0, |\lambda|=2$ (graviton) in the spectrum.

— Massless $\lambda=-1/2$ particle vs $(\frac{1}{2},0)-\psi_\alpha$ Field —

We would like to show now there is no need or emergence of gauge invariance for helicity $\pm 1/2$ fermions which are interpolated by $(1/2,0)$ or $(0,1/2)$ fields. For concreteness, let's focus on $\lambda=-1/2$ and $\psi_\alpha=(1/2,0)$:

(46) $$\langle 0|\,\psi_\alpha(0)\,|\bar{k}\;\lambda=-1/2\rangle_{m=0} \equiv u^\lambda_\alpha(\bar{k}) \overset{?}{=} \qquad \bar{k}^\mu=\bar{E}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$$

Since $j_+=0 \Rightarrow \vec{J}=\vec{J}_-=\frac{\vec{\sigma}}{2} \qquad \vec{K}=i\frac{\vec{\sigma}}{2}$

(47) $$\left.\begin{aligned} W^1 &= \bar{E}(J^1-K^2) = \frac{\bar{E}}{2}\left(\begin{pmatrix}0&1\\1&0\end{pmatrix}-\begin{pmatrix}0&1\\-1&0\end{pmatrix}\right) = \bar{E}\begin{pmatrix}0&0\\1&0\end{pmatrix}\\ W^2 &= \bar{E}(J^2+K^1) = \frac{\bar{E}}{2}\left(\begin{pmatrix}0&-i\\i&0\end{pmatrix}+\begin{pmatrix}0&i\\i&0\end{pmatrix}\right) = \bar{E}\begin{pmatrix}0&0\\i&0\end{pmatrix}\end{aligned}\right\} \boxed{\begin{aligned}W_+&=W^1+iW^2=0\\ W_-&=W^1-iW^2=\begin{pmatrix}0&0\\2&0\end{pmatrix}\bar{E}\end{aligned}}$$

The constraint from rotations give, $\langle 0|\,\psi_\alpha(0)\,|\bar{k}\;\lambda\rangle_{m=0} = (R^{-1})_\alpha{}^\beta\,\langle 0|\,\psi_\beta(0)\,|\bar{k}\;\lambda\rangle_{m=0}\,e^{-i\lambda\theta}$ i.e.

(48) $$\frac{\sigma^3}{2}{}_\alpha{}^\beta\, u^\lambda_\beta(0) = \lambda\, u^\lambda_\beta(0) \Rightarrow \boxed{\begin{aligned}&\lambda=\pm 1/2\\ &u^{-1/2}_\alpha \propto \begin{pmatrix}0\\1\end{pmatrix} \quad u^{+1/2}_\alpha \propto \begin{pmatrix}1\\0\end{pmatrix}\end{aligned}}$$

$$\left(\sigma^3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right)$$

from rotation alone it seems both helicity would be allowed, but

Then, demanding the triviality of $ISO(2)$-translations $T_\alpha{}^\beta=\exp(i\alpha_+W_-+i\alpha_-W_+)$

(49) $$\underline{W_{\pm\,\alpha}{}^\beta\, u^\lambda_\beta = 0} \qquad \Rightarrow \lambda=+1/2 \text{ forbidden } \quad (W_-u^{+1/2}\neq 0)$$

Lesson: • $\psi_\alpha=(1/2,0)$ can interpolate ("destroy") only $\lambda=-1/2$ $m=0$ states
• $\chi^{\dot\alpha}=(0,1/2)$ can interpolate ("destroy") only $\lambda=+1/2$ $m=0$ states

(this second point: either repeat analysis or remember $u^*_{\dot\alpha}\,\varepsilon^{\dot\beta\dot\alpha}=(0,1/2)=\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} \propto \begin{pmatrix}1\\0\end{pmatrix}$. Both $(1/2,0)$ and $(0,1/2)$ transform the same way under rotations: $\frac{\sigma^3}{2}\begin{pmatrix}1\\0\end{pmatrix}=+\frac{1}{2}\begin{pmatrix}1\\0\end{pmatrix} \Rightarrow \lambda=+1/2$)

Q: What is the covariant constraint?

Weyl equation

(50) $$\bar{k}^\mu \bar{\sigma}_\mu = \bar{E}(\sigma^0 + \sigma^3) = E\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \bar{k}^\mu \bar{\sigma}_\mu \cdot u^{-1/2}(\bar{k}) = 0 \Rightarrow \boxed{k^\mu \bar{\sigma}_\mu u^{-1/2}(k) = 0}$$

Like Maxwell, Klein-Gordon also Weyl equation can't be changed: it just express the "kinematical" constraint from Poincaré.

$$\Big[\ u^\lambda_\alpha(k) = \langle 0|\psi_\alpha(0)|k\,\lambda\rangle_{m=0} = \langle 0|\psi_\alpha(0)\,U(L(k,\bar{k}))|\bar{k}\,\lambda\rangle_{m=0} = \Lambda_\alpha{}^\beta(L(k,\bar{k}))\,u^\lambda_\beta(\bar{k})$$

$$\Rightarrow k^\mu \bar{\sigma}_\mu \cdot u^\lambda(k) = k^\mu \bar{\sigma}_\mu \cdot \Lambda_L \cdot u^\lambda(\bar{k}) = k^\mu \Lambda_R \Lambda_L^\dagger \bar{\sigma}_\mu \Lambda_L\, u^\lambda(\bar{k}) = \Lambda_R\, k^\mu L_\mu{}^\nu \bar{\sigma}_\nu u^\lambda(\bar{k})$$

$$= \Lambda_R\, \bar{k}^\mu \bar{\sigma}_\mu u^\lambda(\bar{k}) = 0 \ \Big]$$

General fields: a truely Lorentz-covariant $\psi_A^{(j_-,j_+)}$ has non-vanishing overlap with $m=0$ λ-helicity iff $\lambda = j_+ - j_-$

However, one can easily get around it via gauge invariance

Example: $\psi_{\mu\alpha} = (1/2,1/2)\otimes(1/2,0)$ vs $\lambda = -3/2$

Since $(1/2,1/2)\otimes(1/2,0) = (1,1/2)\oplus\underbrace{(0,1/2)}_{\text{right-handed} = \bar{\sigma}_\mu^{\dot{\alpha}\alpha}\psi^\mu_\alpha}$

Defining the wavefunction as

$$u^\lambda_{\mu\alpha}(\bar{k}) = \langle 0|\psi_{\mu\alpha}(0)|\bar{k}\,\lambda\rangle \longrightarrow \bar{\sigma}^{\mu\dot{\alpha}\alpha} u^\lambda_{\mu\alpha}(0) = 0$$

$$\Big(\ \bar{\sigma}^{\mu\dot{\alpha}\alpha}\langle 0|\psi_{\mu\alpha}(0)|\bar{k}\,\lambda\rangle = \bar{\sigma}^{\mu\dot{\alpha}\alpha}\langle 0|U(R_z)\,\psi_{\mu\alpha}(0)\,U(R_z^{-1})|\bar{k}\,\lambda\rangle\, e^{-i\lambda\theta} =$$

$$= \bar{\sigma}^{\mu\dot{\alpha}\alpha}\langle 0|\psi_{\mu\alpha}(0)|\bar{k}\,\lambda\rangle \cdot \exp\left(i\theta\left(\tfrac{1}{2}-\lambda\right)\right) \quad \text{and } \lambda = -3/2 \neq 1/2\ \Big)$$

While the $\lambda = -3/2$ is associated to

$$\langle 0|\psi_{\mu\alpha}(0)|\bar{k}\,\lambda\rangle_{m=0} = \varepsilon^{-1}_\mu u^{-1/2}_\alpha \quad \text{and its gauge invariance } \varepsilon_\mu \to \varepsilon_\mu + \bar{k}_\mu$$

that give rise to $\psi_{\mu\alpha} \to \Lambda_\mu{}^\nu \Lambda_\alpha{}^\beta(\psi_{\nu\beta} + \partial_\nu \Omega_\beta)$ with Ω_β a left-handed fermion.